Binomial Models

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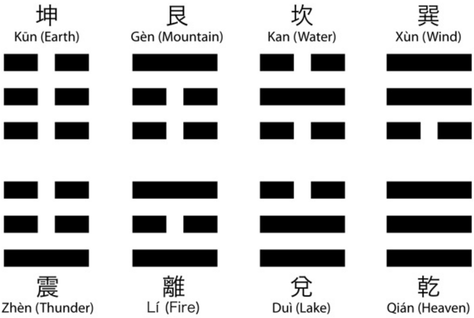
Applications of Randomness?

→ I Ching (Yi Jing)

→ Casting of lot

Source: CARM



Source: brainpickings.org

Binary Stochastic Processes

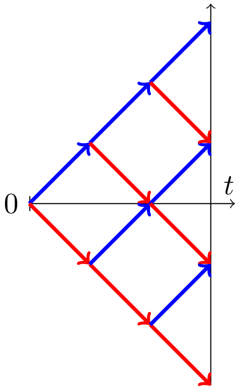
- Blank coin with blue dot on one side and red dot on the other side.
- Equally likely for either blue or red to turn up
- Random variable $C(\omega)$ is a mapping into numbers.

$$C(\omega) = \begin{cases} 1, & \text{if } \omega \text{ is blue;} \\ -1, & \text{if } \omega \text{ is red.} \end{cases}$$

3-Period Binomial Tree

$D_1\ D_2\ D_3 \quad U_1\ D_2\ D_3 \quad D_1\ U_2\ D_3 \quad U_1\ U_2\ D_3$

$D_1\ D_2\ U_3 \quad U_1\ D_2\ U_3 \quad D_1\ U_2\ U_3 \quad U_1\ U_2\ U_3$



One-Dimensional Random Walk

- A model of random walk S_t

$$S_t = S_{t-1} + C_t.$$

- Sum of up-down moves:

$$S_t = S_0 + \sum_{i=1}^{t} C_i$$

- Mean of C_t

$$\mathbb{E}(C_t) = \frac{1}{2} \times 1 + \frac{1}{2} \times (-1) = 0$$

- Variance of C_t

$$\mathbb{V}(C_t) = \frac{1}{2} \times (1-0)^2 + \frac{1}{2} \times (-1-0)^2 = 1$$

Mean-Variance Analysis of 1-D Random Walk

- Unconditional mean of random walk

$$\mathbb{E}(S_t) = \mathbb{E}\left(S_0 + \sum_{i=1}^{t} C_i\right) = S_0 + \sum_{i=1}^{t}\mathbb{E}(C_i) = S_0$$

- Unconditional variance of random walk

$$\mathbb{V}(S_t) = \mathbb{V}\left(S_0 + \sum_{i=1}^{t} C_i\right) = \mathbb{V}\left(\sum_{i=1}^{t} C_i\right)$$

- Since C_i and C_j are independent for $i \neq j$,

$$\mathbb{V}\left(\sum_{i=1}^{t} C_i\right) = \sum_{i=1}^{t}\mathbb{V}(C_i) = \sum_{i=1}^{t} 1 = t.$$

Conditional Mean of 1-D Random Walk

- At time t, S_{t-1} is known. Given S_{t-1}, what is the expected value of S_t?

$$\begin{aligned}\mathbb{E}\big(S_t\big|S_{t-1}\big) &= \mathbb{E}\big(S_{t-1} + C_t\big|S_{t-1}\big)\\ &= \mathbb{E}\big(S_{t-1}\big|S_{t-1}\big) + \mathbb{E}\big(C_t\big|S_{t-1}\big)\\ &= S_{t-1}.\end{aligned}$$

- What is the interpretation of this conditional mean?

Answer: ______________________________

Martingale

- Every coin toss C_i is independent of every other.
- Intuitively, the drunken man has no memory of where he has been before.
- Even if all information of the past is provided, still

$$\mathbb{E}\big(S_t \big| S_{t-1}, S_{t-2}, \ldots, S_0\big) = S_{t-1}.$$

A fundamental theorem of financial mathematics

A financial market is viable (i.e., no risk-free arbitrage opportunity) if and only if there exists a probability measure under which the prices are martingales.

Conditional Variance of 1-D Random Walk

- Given S_{t-1}, what is the variance of S_t?

$$\begin{aligned}\mathbb{V}\left(S_t \middle| S_{t-1}\right) &= \mathbb{V}\left(S_{t-1} + C_t \middle| S_{t-1}\right)\\ &= \mathbb{V}\left(S_{t-1} \middle| S_{t-1}\right) + \mathbb{V}\left(C_t \middle| S_{t-1}\right)\\ &= 0 + 1\\ &= 1\end{aligned}$$

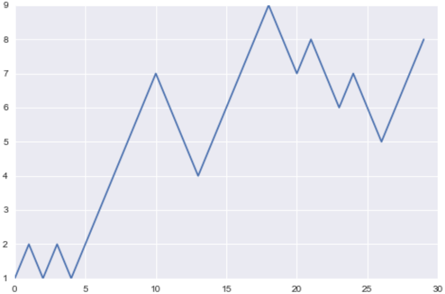
- What is the interpretation of this conditional variance?

Answer: ____________________________________

Where is the Random Walker After n Steps?

- Let n_+ be the number of upward moves, and n_- the number of downward moves.
- By definition, the total must add up: $n_+ + n_- = n$.
- Also,

$$S_n = S_0 + \sum_{i=1}^{n_+} \mathsf{U}_i + \sum_{j=1}^{n_-} \mathsf{D}_1 = S_0 + n_+ - n_-.$$



Multiplicative Random Walk

- Positive random variable Λ_t

$$\Lambda_t(\omega) = \begin{cases} u > 1, & \text{if } \omega \text{ is blue;} \\ d < 1, & \text{if } \omega \text{ is red.} \end{cases}$$

- Multiplicative random walk

$$S_t = S_{t-1}\Lambda_t$$

- Up to time t from time 0, the price of the asset is obtained as

$$S_t = S_0 \prod_{i=1}^{t} \Lambda_i. \tag{1}$$

Recombinant Binomial Tree

- The order of movements does not matter.

$$S_t = udS_{t-2} = duS_{t-2} = S_{t-2}.$$

- To make the binomial tree recombinant, you must have

$$u = \frac{1}{d}.$$

- Otherwise, for 100 periods, the number of end nodes S_T is $2^{100} \approx 10^{30}$.
- At each node, you compute $S_T - K$ for a call option. How long does it take on the fastest supercomputer? Given 33.86 Pflop/sec, which is 33.86×10^{15}, the time taken is

$$\frac{2^{100}}{33.86 \times 10^{15}} = 3.74 \times 10^{13} \text{ seconds} \approx \text{ 1,186,339 years!}.$$

One-Period Case

- At time 0, the underlying asset price is S_0,
- At time 1,

$$S_1(\omega) = \begin{cases} uS_0, & \text{if } \omega \text{ is blue;} \\ dS_0, & \text{if } \omega \text{ is red.} \end{cases}$$

- At time 0, construct a portfolio consisting a long position x_0 units of risky asset at S_0 per unit, and a short position in a call option at c_0 struck at K, for which $dS_0 < K < uS_0$. The value (opposite of cash flow) of the portfolio is

$$V_0 = x_0 S_0 - c_0.$$

- At time 1, the value of the portfolio is
  - If ω is blue,
  $$V_1^+ = x_0 u S_0 - (uS_0 - K).$$
  - If ω is red,
  $$V_1^- = x_0 d S_0.$$

Toy-Model Pricing

- To remove uncertainty and therefore risk, set $V_1^+ = V_1^-$

$$V_1^+ = x_0 u S_0 - (uS_0 - K) = x_0 d S_0 = V_1^-.$$

- Solving for x_0,

$$x_0 = \frac{uS_0 - K}{(u-d)S_0}. \qquad (2)$$

- Therefore, the portfolio value at time 1 is also known at time 0, and we have

$$V_1 = V_1^+ = V_1^- = \frac{d(uS_0 - K)}{u-d} = \frac{S_0 - dK}{u-d}.$$

Toy-Model Pricing (Cont'd)

- Since there is no uncertainty, by the first Principle of $\mathbb{Q}$F, the present value of the portfolio at time 0 must be discounted by the risk-free rate r_0 as follows:

$$V_0 = e^{-r_0} V_1 = \frac{e^{-r_0} d(uS_0 - K)}{u - d}.$$

We shall call e^{-r_0} the risk-free discount factor. But the portfolio value at time 0 is $V_0 = x_0 S_0 - c_0$

- Substituting in x_0 from (2), we find that the value c_0 of the call option should be

$$c_0 = x_0 S_0 - \frac{e^{-r_0} d(uS_0 - K)}{u - d} = \frac{1 - e^{-r_0} d}{u - d}(uS_0 - K). \quad (3)$$

Risk Neutrality

- In toy-model pricing, the probability p of upward movement by the underlying asset is not needed.
- By definition, the risk-free rate r_0 is indifferent to the up-or-down outcome. If the dollar value corresponding to the risky asset S_0 is invested in the risk-free bond, the value of this bond will become $e^{r_0} S_0$ at time 1 for sure.
- On the other hand, with probability p, the upward outcome is uS_0, whereas the downward outcome dS_0 occurs with a probability of $1 - p$.
- Being risk neutral means that the expected return of the risky asset is the risk-free r_0.

$$\frac{\mathbb{E}_0^{\mathbb{Q}}(S_1)}{S_0} = e^{r_0} \Longrightarrow uS_0 \times p + dS_0 \times (1 - p) = e^{r_0} S_0. \qquad (4)$$

Risk-Neutral Probability

- Solving Equation (4) for p, we obtain the probability p of upward movement:

$$p = \frac{e^{r_0} - d}{u - d}. \quad (5)$$

- The probability q of downward movement is

$$q = 1 - p = \frac{u - e^{r_0}}{u - d}. \quad (6)$$

- Since the probability is positive, it must be that

$$d < e^{r_0} < u. \quad (7)$$

Numerical Illustration (Thanks for the Mid-Term Feedback!)

- Suppose the up factor u is 1.25. That is 25% return.
- For the binomial tree to be recombinant, the down factor is $d = 1/u$ = 1/1.25 = 0.80, i.e., a decline of 20%.
- Let r_0 be 1%, hence $e^{0.01}$ = 1.01.
- The risk-neutral probability of upward movement is

$$p = \frac{1.01 - 0.80}{1.25 - 0.80} = 46.67\%$$

- The risk-neutral probability of downward movement is

$$q = 1 - p = \frac{1.25 - 1.01}{1.25 - 0.80} = 53.33\%.$$

Risk-Neutral Probability in Call and Put Prices

- Applying the put-call parity, you can compute from (3) to obtain
$$p_0 = \frac{ue^{-r_0} - 1}{u - d}(K - dS_0). \tag{8}$$
- The risk-neutral probability (5) is embodied in the pricing formulas for call and put options, (3) and (8), respectively. To see it, we rewrite these pricing formulas as
$$c_0 = \frac{e^{r_0} - d}{u - d} e^{-r_0}(uS_0 - K), \qquad p_0 = \frac{u - e^{r_0}}{u - d} e^{-r_0}(K - dS_0).$$
- It is easy to find that
$$c_0 = p\, e^{-r_0}(uS_0 - K), \qquad p_0 = q\, e^{-r_0}(K - dS_0).$$

Numerical Examples

- From the earlier example, we have $p = 46.67\%$, $q = 53.33\%$, $u = 1.25$, and $d = 0.80$. Also the risk-free rate is $r_0 = 1\%$.
- Suppose S_0 = \$5, and the strike price K is \$5.5.
- The call price is

$$c_0 = 0.4667 \times e^{-0.01} \times \left(1.25 \times \$5 - \$5.25\right) = \$0.46.$$

- The put price is

$$p_0 = 0.5333 \times e^{-0.01} \times \left(\$5.25 - 0.8 \times \$5\right) = \$0.66.$$

Revisiting the First Principle of $\mathbb{Q}$F

- We denote the value of the call option at time 1 as $c_1^+ := uS_0 - K$ for the up state, and that for the down state as $c_1^- = 0$.
- In general c_1^- may not be zero when the options are deep in the money.
- With the risk-neutral probability p, we invoke the notion of expected future value of c_1, which is denoted by $\mathbb{E}_0^{\mathbb{Q}}\big(c_1\big)$, and we write

$$c_0 = e^{-r_0}\big(pc_1^+ + (1-p)c_1^-\big) = e^{-r_0}\mathbb{E}_0^{\mathbb{Q}}\big(c_1\big). \tag{9}$$

$$p_0 = e^{-r_0}\big(pp_1^+ + (1-p)p_1^-\big) = e^{-r_0}\mathbb{E}_0^{\mathbb{Q}}\big(p_1\big). \tag{10}$$

- Under the risk-neutral probability p, the fair price of option today is the expected value of its future payoff discounted by the risk free rate.

Delta Hedging

- To make the portfolio value invariant to both outcomes, i.e., remove the uncertainty arising from the randomness of coin tossing, the long position in the stock is required to hedge against a short position in the call option.
- Delta-hedging ratio

$$x_0 = \frac{c_1^+ - c_1^-}{S_1^+ - S_1^-}, \tag{11}$$

where S_1^+ is the value S_1 of the underlying asset at time 1 in the up state and S_1^- is the value for the down state.
- Since $S_1^+ = uS_0$, $S_1^- = dS_0$, $c_1^+ = uS_0 - K$, and $c_1^- = 0$, the delta-hedging ratio (11) indeed yields the same result as (2):

$$x_0 = \frac{uS_0 - K}{(u-d)S_0}.$$

One-Period Change of Wealth

- The initial cash amount or wealth is denoted by W_0. You buy x_0 shares of the underlying asset of the call option at the known price of S_0. The left-over cash is

$$M_0 := W_0 - x_0 S_0, \tag{12}$$

which is invested in the risk-free money market.

- One period later, the wealth W_1 will become

$$W_1 = \begin{cases} W_1^+ = x_0 S_1^+ + e^{r_0} M_0, & \text{blue outcome;} \\ W_1^- = x_0 S_1^- + e^{r_0} M_0, & \text{red outcome.} \end{cases}$$

It is important to note that the two outcomes are anticipated at time 0.

Replication of Call's Payoff

- The replication approach is to make the cash flow W_1 at time 1 equal the option's payoff c_1:

$$W_1 = x_0 S_1 + e^{r_0} M_0 = c_1.$$

- To achieve this replication, using definition (12), we first rewrite the cash flow W_1 of the portfolio as

$$\begin{aligned} W_1 &= e^{r_0} W_0 + x_0 \big(S_1 - e^{r_0} S_0 \big) \\ &= e^{r_0} \big(W_0 + x_0 (e^{-r_0} S_1 - S_0) \big). \end{aligned}$$

- We express the replication by matching each of the possible outcome:

$$\begin{aligned} W_0 + x_0 \left(e^{-r_0} S_1^+ - S_0 \right) &= e^{-r_0} c_1^+, \\ W_0 + x_0 \left(e^{-r_0} S_1^- - S_0 \right) &= e^{-r_0} c_1^-. \end{aligned}$$

Replication of Call's Payoff (Cont'd)

- To arrive at this equalization, we need to find the values of x_0 and W_0.
- Multiplying the upward outcome by p and the downward outcome by $1-p$, and after adding them together, we obtain

$$W_0+x_0\left(e^{-r_0}\left(pS_1^+ + (1-p)S_1^-\right) - S_0\right) = e^{-r_0}\left(pc_1^+ + (1-p)c_1^-\right).$$

- Because of (4), the sum of the two terms, $pS_1^+ + (1-p)S_1^-$, is equal to $e^{r_0}S_0$.
- Consequently,

$$W_0 = e^{-r_0}\left(pc_1^+ + (1-p)c_1^-\right).$$

In view of (9), W_0 is in fact the value of the option c_0 at time 0.

Multi-Period Generalization

- For each node on the binomial tree that is not an ending node, the cash flow W_t at time t is

$$W_t = \begin{cases} W_t^+ = x_{t-1} S_t^+ + e^{r_0} M_{t-1}, & \text{blue outcome;} \\ W_t^- = x_{t-1} S_t^- + e^{r_0} M_{t-1}, & \text{red outcome.} \end{cases}$$

- The money market account M_t maturing at time $t+1$ is

$$M_t = W_t - x_t S_t, \qquad \text{for } t = 0, 1, 2, \ldots, T-1.$$

It is the fund left (or needed if M_t is negative) after taking a long position of x_t in the risky underlying asset at the price of S_t.

Multi-Period Generalization (Cont'd)

- The delta-hedging ratio at time t for $t+1$ is

$$x_t = \frac{c_{t+1}^{+} - c_{t+1}^{-}}{S_{t+1}^{+} - S_{t+1}^{-}} = \frac{c_{t+1}^{+} - c_{t+1}^{-}}{(u-d)S_t}. \quad (13)$$

- Moreover, the risk-neutral pricing model is

$$c_t = e^{-r_0}\mathbb{E}_t\big(c_{t+1}\big). \quad (14)$$

Proposition

- $W_t = c_t$ from time 0 up to time $T-1$.
- For each t of the binomial tree, the risk-neutral valuation of a pair of future payoffs is

$$c_t = e^{-r_0}\big(pc_{t+1}^{+} + (1-p)c_{t+1}^{-}\big) = e^{-r_0}\mathbb{E}\big(c_{t+1}\big). \quad (15)$$

Proof of Proposition by Induction

- Assume that $W_t = c_t$ is true and show that $W_{t+1} = c_{t+1}$ also holds.
- First, replication means that for the up state, i.e., $S_{t+1}^{+} = uS_t$,

$$W_{t+1}^{+} = x_t S_{t+1}^{+} + e^{r_0}\left(W_t - x_t S_t\right) = e^{r_0} W_t + x_t S_t (u - e^{r_0}), \tag{16}$$

- Substituting the delta-hedging ratio x_t (13) into (16), we obtain

$$\begin{aligned} W_{t+1}^{+} &= e^{r_0} W_t + \frac{\left(c_{t+1}^{+} - c_{t+1}^{-}\right)\left(u - e^{r_0}\right)}{u - d} \\ &= e^{r_0} W_t + (1-p) c_{t+1}^{+} - (1-p) c_{t+1}^{-}. \end{aligned}$$

Proof of Proposition by Induction (Cont'd)

- In view of (14) and the forward induction assumption that $W_t = c_t$, we have

$$e^{r_0} W_t = e^{r_0} c_t = \mathbb{E}_t^{\mathbb{Q}}\big(c_{t+1}\big) = p c_{t+1}^{+} + (1-p) c_{t+1}^{-}.$$

  Hence,

$$W_{t+1}^{+} = p c_{t+1}^{+} + (1-p) c_{t+1}^{+} = c_{t+1}^{+}.$$

- Second, using the same method, you can also show that

$$W_{t+1}^{-} = p c_{t+1}^{-} + (1-p) c_{t+1}^{-} = c_{t+1}^{-}.$$

- Accordingly, if $W_t^{\pm} = c_t^{\pm}$, then $W_{t+1}^{\pm} = c_{t+1}^{\pm}$. We have already shown that $t = 0$ is true, i.e., $W_0 = c_0$. At time $t = 1$, $W_1 = c_1$ must also be true, and so on. Thus, the proof by forward induction is complete.

Connection of Volatility to the Up and Down Factors

- The up and down factors depend on the rate of variance σ^2 of the underlying asset's return.
- The rate of variance σ^2 quantifies the degree of fluctuation exhibited by the return on the underlying asset.
- The variance for a time period t is $\sigma^2 t$, and the volatility is its square root $\sigma\sqrt{t}$. We set

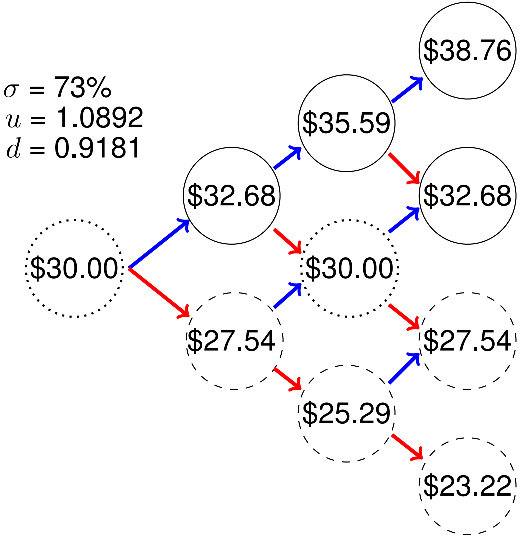
$$u = e^{\sigma\sqrt{t}}, \qquad \text{and} \qquad d = e^{-\sigma\sqrt{t}}.$$

- Note from (7) that the risk-free factor $e^{r_0 t}$ must be smaller than the up factor u, i.e., $e^{r_0 t} < e^{\sigma\sqrt{t}}$. It follows that the time interval t of each period must satisfy

$$\sqrt{t} < \frac{\sigma}{r_0}.$$

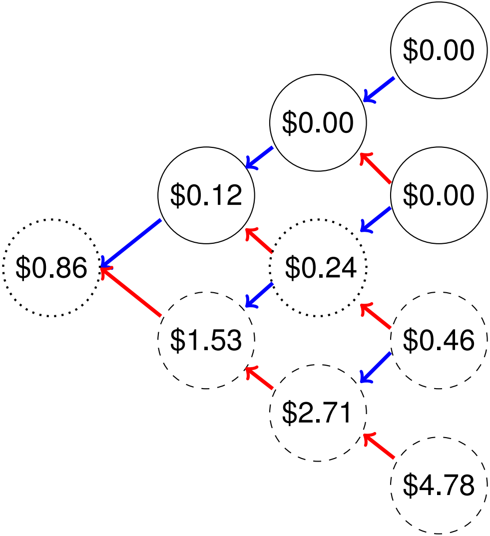
A Numerical Example of Binomial Option Pricing

- Asset prices for all nodes
- S_0 = ________
- Put option's days to maturity = 15 days
- Since $N = 3$, each period is 15/3 = 5 days
- 5 days is t = 5/365 = 1/73 years
- risk-free rate r_0 = 0.25%



Put Option Prices

- Strike price = $28
- Upward probability p = 47.89%



Random Walk Once More

- The multiplicative random walk (1) in the context of multi-period binomial model takes the following form:

$$S_t = u^k d^{t-k} S_0.$$

- What is the probability of reaching $S_t = u^2 d S_0$?
- Answer: The binomial probability for the random number $\widetilde{N}$ of obtaining the blue dot on top when tossing the blue-red coins T times. In this case, $\widetilde{N} = 2$ and $T = 3$. Therefore the probability is

$$\mathbb{P}\big(3; \widetilde{N} = 2\big) = \binom{3}{2} p^2 (1-p)^1 = 3p^2(1-p).$$

Recall Your Pre-U Math

- In general, the binomial probability of the number of "successes" in flipping the blue-red coin, which turns up the blue dot is

$$\mathbb{P}\big(t; \widetilde{N} = k\big) = \binom{t}{k} p^k (1-p)^{t-k}, \tag{17}$$

where the binomial coefficient is

$$\binom{t}{k} := \frac{t!}{k!(t-k)!}.$$

- Here $\widetilde{N}$ is a random variable, as the number of successes is uncertain before the t tosses are completed.

Applying Your Pre-U's Binomial Theorem

- With the probability mass function (17) of the random variable $\widetilde{N}$ for the process of flipping the blue-red coin t times, we can compute the expected value at time 0 of $\widetilde{S}_t$ given S_0 as follows:

$$\mathbb{E}_0\left(\widetilde{S}_t\right) = \sum_{k=0}^{t} u^k d^{t-k} S_0 \mathbb{P}\left(t; \widetilde{N} = k\right).$$

- Interestingly, we find that

$$\begin{aligned}\mathbb{E}_0\left(\widetilde{S}_t\right) &= \sum_{k=0}^{t} u^k d^{t-k} S_0 \binom{t}{k} p^k (1-p)^{t-k} \\ &= S_0 \sum_{k=0}^{t} \binom{t}{k} (up)^k \big(d(1-p)\big)^{t-k} \\ &= S_0 \big(up + d(1-p)\big)^t.\end{aligned}$$

First Principle of $\mathbb{Q}$F Once Again

- Moreover, using the one-period risk-neutral probability p, i.e., (5), we obtain

$$\mathbb{E}_0^{\mathbb{Q}}\big(\widetilde{S}_t\big) = S_0\big((u-d)p+d\big)^t = S_0\, e^{r_0 t}.$$

The reason for using (5) is that each time period we consider here is one unit of time.

- To gain further insight, notice that

$$up + d(1-p) = (u-d)p + d = e^{r_0 t}$$

- Accordingly, under the single-period risk-neutral probability p in (5), the average gross return over one period is simply the forward factor e^{r_0}.
- Multi-period generalization

$$\boxed{S_0 = e^{-r_0 t}\,\mathbb{E}_0\left(\widetilde{S}_t\right).} \qquad (18)$$

Large-Scale Binomial Probability

- Define a probability mass function $B(x)$ of a discrete variable x:

$$B(x) := \mathbb{P}\big(T; \widetilde{N} = x\big) = \frac{T!}{x!(T-x)!} p^x (1-p)^{T-x}. \qquad (19)$$

  It is the binomial probability of x number of successes in getting the blue dot on top out of T tosses.
- The large number T is the result of slicing the time period t into many tiny pieces of size δt, which is a very short duration. We write

$$T = \frac{t}{\delta t}.$$

- It is noteworthy that T can be made arbitrarily large when δt is set at an arbitrarily small number. Even so, their product, i.e., $T \times \delta t$ is a non-zero finite number t.

Mean and Variance

- Under the risk-neutral probability p, the mean of the random variable Λ_i in (1) is, as computed before,

$$\mathbb{E}_0^{\mathbb{Q}}\left(\Lambda_i\right) = up + d(1-p) = e^{r_0\delta t}, \qquad \text{for each } i$$

- The variance of Λ_i is

$$\begin{aligned}
\mathbb{V}^{\mathbb{Q}}\left(\Lambda_i\right) &= \left(u - e^{r_0\delta t}\right)^2 p + \left(d - e^{r_0\delta t}\right)^2 (1-p) \\
&= p\left(\left(u - e^{r_0\delta t}\right)^2 - \left(d - e^{r_0\delta t}\right)^2\right) + \left(d - e^{r_0\delta t}\right)^2 \\
&= \left(e^{r_0\delta t} - d\right)\left(u + d - 2e^{r_0\delta t}\right) + \left(e^{r_0\delta t} - d\right)^2 \\
&= p(1-p)(u-d)^2.
\end{aligned}$$

- Given that δt is small, the variance $\mathbb{V}\left(\Lambda_i\right)$ is well approximated by

$$\mathbb{V}^{\mathbb{Q}}\left(\Lambda_i\right) \approx 4p(1-p)\sigma^2\delta t.$$

Takeaways

- Random walk, though simple, is a great model with which to think about randomness and probability.
- Binomial trees are useful models for pricing options.
- up factor, down factor, risk-free rate, and risk neutral probability are closely related.
- Up and down factors are dependent on the volatility of the underlying asset.
- In the asymptotic limit, the binomial model converges to the Black-Scholes formula for pricing European options.

Week 9 Assignment

1. Given the parametrization $u = e^{\sigma t}$ and risk-free rate r_0 for a recombinant binomial tree, show that the risk-neutral upward probability p can be well approximated as

$$p = \frac{e^{r_0 t} - d}{u - d} \approx \frac{1}{2}\left(1 + \frac{\left(r_0 - \frac{1}{2}\sigma^2\right)\sqrt{t}}{\sigma}\right).$$

2. Given t = 1/73, r_0 = 0.25%, and σ = 73% as in Slide 32, how good is the above approximation of p compared to the exact value in Slide 33?
3. Suppose $\delta t = t$, i.e., no splicing of the period. Using the same parameter values in Problem (2), compute the variance of Λ_i under the risk-neutral probability p.
4. Given the binomial price tree in Slide 32, price the ATM call option.

Week 9 Additional Exercises

1. For the one-dimensional random walk with p being the upward move by $+1$ and q being the downward move by -1, what is the probability for the drunken man to be at $S_n = x > 0$ after n steps?
(Hint: Let $f(m)$ be the probability that $x = m$ is ever reached. Then $f(m+1) = f(m)f(1)$.)
2. For the one-dimensional random walk in Problem 1, Let $m > 0$ and $n > 0$. What is the probability $g(m, n)$ of reaching the point $x = +m$ before $x = -n$?
3. A stock analyst with some special powers is able to guess correctly the flip of a coin with 60% probability. He starts with two million dollars, and plays a game of guessing the toss with a sovereign wealth fund, which has almost infinite amount of money. What is the probability that the stock analyst will ultimately lose all the money?